A result on the coupled fixed point theorems in C^*-algebra valued b-metric spaces

Özen Özer*
Department of Mathematics
Faculty of Science and Arts
Kırklareli University
Kırklareli, 39100
Turkey
ozenozer39@gmail.com

Saleh OMRAN
Department of Mathematics
Faculty of Science
Taif University
Taif, KSA
and
Department of Mathematics
South Valley University
Qena
Egypt
salehomran@yahoo.com

Abstract. The aim of this paper is to establish a new Coupled Fixed Point Theorems for C^*-algebra valued b-metric spaces. As an application of our result, we discuss the existence and uniqueness results for Couple Fixed Point Theorem in C^*-algebra valued b-metric spaces. We also give conclusion to demonstrate our result.

Keywords: fixed point theory, C^*-algebras, b- metric spaces.

1. Introduction

In Blackadar's book [1], he gave many important results and properties of the theory of C^*-algebras and Von Neumann Algebras. Huang and Zhang [7] introduced the concept of cone metric space which generalized the concept of the real valued metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions.Ma and co-authors obtained important results for fixed point theorems on both C^*-algebra valued metric space and C^*-algebra valued b-metric space in [9],[10] respectively.In [12] and [13], we considered C^*- algebras valued metric spaces and proved certain fixed-point theorem for a single valued

*. Corresponding author

mapping in such spaces. In [3], authors introduced the theory of common fixed point theorems on metric spaces.

Subsequently, some other authors studied the existence of fixed points of self mappings satisfying a contractive type condition. The author and et al introduced in [17] the notation of the Quaternion valued metric spaces as well as contraction mapping.For getting more information about the general theory, you can look valuable books [4], [11], [14] and [15] as well as significant papers [16],[8],[2],[5] and [6].

Here, we introduce a new coupled fixed point theorems for C^*-algebra b-metric spaces with some examples.

2. Preliminaries

Basic notions and facts which play a central role in the present paper for C^*-algebra valued b-metric space will be given in this section.

2.1 C^*-algebra and positive cone

Definition 2.1. A C^*-algebra $\mathcal{A}$ is a complex Banach algebra with a conjugate-linear involution $* : \mathcal{A} \longrightarrow \mathcal{A}$, such that

$$(x^*)^* = x,\ (xy)^* = y^*x^*,\ ,(x+y)^* = x^* + y^*\ , \|x^*x\| = \|x\|^2,$$

for all x, y in $\mathcal{A}$. The C^*-condition $\|x^*x\| = \|x\|^2$ implies that the involution is an isometry in the sense that $\|x^*\| = \|x\|$ for all x in $\mathcal{A}$.

A C^*-algebra is called unital if it possesses a unit. It follows easily that $\|1\| = 1$. In general C^*-algebra is non-commutative. We assume the unital C^*-algebras in our work.

Definition 2.2. A *-homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ between two C^*-algebras is a homomorphism such that $\varphi(x^*) = \varphi(x)^*$ for all $x \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are unital, then φ is called unital if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. An isomorphism between two C^*-algebras is a bijective *-homomorphism.

It is an important basic fact that every $*$-homomorphism between C^*-algebras is continuous. If two C^*-algebras are isomorphic, then they are automatically isometric.

Example 2.3. Let $\mathcal{H}$ be an arbitrary Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ with the operator norm $\|x\| = sup\{\|x\xi\|,\ \xi \in \mathcal{H}, \|\xi\| = 1\}$ for each $x \in \mathcal{B}(\mathcal{H})$ and the involution given by the map which assigns $x \in \mathcal{B}(\mathcal{H})$ its adjoint is a C^*-algebra.

Sub-C^*-algebra of a C^*-algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$ which is a C^*-algebra with respect to the operations given on $\mathcal{A}$. Let S be a subset of a C^*-algebra $\mathcal{A}$. Then the sub-C^*-algebra generated by S, denoted by $C^*(S)$, is the smallest sub-C^*-algebra of $\mathcal{A}$ that contains S.

The following fundamental results were proved by Gelfand, Naimark and Segal [**?**]:

Every commutative C^*-algebra is isomorphic to $C_0(X)$ for some locally compact space X. Furthermore, every C^*-algebra is isomorphic to a sub-algebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

2.2 Positivity in C^*-algebra

Definition 2.4. A self adjoint element in C^*-algebra is called positive if $\sigma(a) \in \mathcal{R}^+$. If a is positive, we write $a \geq 0$ and positive cone $\mathcal{A}_+$ is represented by $\mathcal{A}_+ = \{a \in \mathcal{A},\ a \geq 0\}$.

Lemma 2.5. Suppose that $\mathcal{A}$ is a C^*-algebra:

(a) If $a \in \mathcal{A}$ is normal, then $a^*a \geq 0$;

(b) If $a \in \mathcal{A}$ is self-adjoint and $\|a\| \leq 1$, then $a \geq 0$;

(c) If $a, b \in \mathcal{A}_+$ then $a + b \in \mathcal{A}_+$;

(d) $\mathcal{A}_+$ is closed in $\mathcal{A}$;

(e) If $b, c \in \mathcal{A}'_+$, then $b \leq c$ implies that $a^*ba \leq a^*ca$;

(f) If $b_n \subset \mathcal{A}$ and $\lim_{n\to\infty} b_n = 0_{\mathcal{A}}$, then $\lim_{n\to\infty} a^*b_na = 0_{\mathcal{A}}$ for any $a \in \mathcal{A}$.

For a given $a, b \in \mathcal{A}_+$, we denote $a \leq b$ if $a - b \geq 0$, $\mathcal{A}_+$ becomes a partially ordered vector space. Furthermore in a unital C^*-algebra, for all $a, b \in \mathcal{A}$, $0 \leq a \leq b$ implies that $\|a\| \leq \|b\|$, so the positive cone in a C^*-algebra is automatically normal.

Lemma 2.6. Suppose that $\mathcal{A}$ is unital C^*-algebra with a unit I:

(1) If $a \in \mathcal{A}$ with $\|a\| \leq \frac{1}{2}$ then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$;

(2) Suppose that $a, b \in \mathcal{A}$ with $a, b \geq \theta$ and $ab = ba$, then $ab \geq \theta$.

2.3 C^*-algebras valued b-metric space

C^*-algebras valued b-metric space introduced by [**?**] and studied self mapping fixed point theorem, here we will introduce some other examples and the relation between C^*-algebras valued b-metric space and Fixed Point Theory. Moreover, we will introduce the common fixed point theory in this setting.

Definition 2.7. Let X be a non-empty set and $\mathcal{A}$ is a C^*-algebra, $\mathcal{A}_+$ be its positive cone. A C^*-algebra valued b-metric space has a function $d : X \times X \longrightarrow \mathcal{A}_+$ defined on X such that for any $x, y, z \in X$ and $a \in \mathcal{A}'_+$:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq a(d(x, y) + d(y, z))$.

From the the definition its automatically that $d(x, y) \geq 0,\ d(x, x) \geq 0$.

Definition 2.8. Let $(\mathrm{X}, \mathcal{A}, d)$ be a C^*-algebra valued b-metric space and $x_n \subset X$ is a sequence in X. If $x \in X$ and $\varepsilon > 0$ there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$ then (x_n) is called a convergent sequence in X to x and denote it by $\lim_{n\to\infty} x_n = x$

Moreover, if for any $\varepsilon > 0$ there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$ then (x_n) is called a cauchy sequence in X.

Definition 2.9. The tripled $(\mathrm{X}, \mathcal{A}, d)$ is a complete C^*-algebra valued b-metric space if every cauchy sequence is convergent.

Definition 2.10. Let $(\mathrm{X}, \mathcal{A}, d)$ be a C^*-algebra valued b-metric space. The mapping $T : X \longrightarrow X$, is called contractive mapping on x if there is an $a \in \mathcal{A}$ such that $\|a\| < 1$ and satisfy

$$d(T(x), T(y)) \leq a^* d(x, y) a$$

for $x, y \in X$.

Example 2.11. Let $X = [-1, 1]$ and $\mathcal{A} = M_{2\times 2}(R)$ with $\|A\| = \max(|a_1|, |a_2|, |a_3|, |a_4|)$ where a_i's are the entries of the matrice $A \in M_{2\times 2}(R)$. Then $(\mathrm{X}, \mathcal{A}, d)$ is a C^*-algebra valued b-metric space, where

$$d(x, y) = \begin{pmatrix} |x-y|^p & 0 \\ 0 & |x-y|^p \end{pmatrix}$$

for $p > 0$ and

$$|x - z|^p \leq 2^p I(|x - y|^p + |y - z|^p)$$

as well as partial ordering on $\mathcal{A}$ is given as

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \geq \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \Leftrightarrow a_i \geq b_i$$

for $i = 1, 2, 3, 4$.

So the condition (3) is in the form satisfies

$$d(x, y) \leq 2^p I(\begin{pmatrix} |x-z|^p & 0 \\ 0 & |x-z|^p \end{pmatrix} + \begin{pmatrix} |z-y|^p & 0 \\ 0 & |z-y|^p \end{pmatrix}),$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Main theorem and results

Theorem 3.1. Let $(\mathrm{X}, \mathcal{A}, d)$ be a complete C^*-algebra valued b-metric space.We suppose that the mapping $T : X \times X \longrightarrow X$ satisfies

$$d(T(x,y), T(u,v)) \leq a[Ad(x,u)A^* + Bd(y,v)B^*], \tag{1}$$

for $x, y, u, v \in X$ and $A, B \in \mathcal{A}$, such that $\|A\| \leq \frac{1}{2}$ and $\|B\| \leq \frac{1}{2}$. Then T has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ such that

$$x_1 = T(x_0, y_0), y_1 = T(y_0, x_0) \tag{2}$$
$$x_2 = T(x_1, y_1), y_1 = T(y_1, x_1) \tag{3}$$
$$. \tag{4}$$
$$. \tag{5}$$
$$. \tag{6}$$
$$x_m = T(x_{m-1}, y_{m-1}), y_1 = T(y_{m-1}, x_{m-1}) \tag{7}$$
$$x_{m+1} = T(x_m, y_m), y_1 = T(y_m, x_m) \tag{8}$$

From (1) and (2) we have

$$d(x_{n+1}, x_n) = d(T(x_n, y_n), T(x_{n-1}, y_{n-1}) \leq a(Ad(x_n, x_{n-1})A^* + Bd(y_n, y_{n-1})B^*)$$

and similarly

$$d(y_{n+1}, y_n) = d(T(y_n, x_n), T(y_{n-1}, x_{n-1}) \leq a(Ad(y_n, y_{n-1})A^* + Bd(x_n, x_{n-1})B^*).$$

So, we get

$$\begin{aligned}
&d_{nn} \leq a(Ad(x_n, x_{n-1})A^* + Bd(y_n, y_{n-1})B^* + Ad(y_n, y_{n-1})A^* + Bd(x_n, x_{n-1})B^*) \\
&= a(A+B)d(x_n, x_{n-1})(A^* + B^*) + a(A+B)d(y_n, y_{n-1})(A^* + B^*) \\
&= a(A+B)[d(x_n, x_{n-1}) + d(y_n, y_{n-1})](A^* + B^*) \\
&= a(A+B)d_{(n-1)(n-1)}(A^* + B^*)
\end{aligned}$$

by taking $d_{nn} = d(x_{n+1}, x_n) + d(y_{n+1}, y_n)$ for all $n \geq 0$.

In general, we have

$$\begin{aligned}
&d_{nn} \leq a(A+B)d_{(n-1)(n-1)}(A^* + B^*) \\
&\leq a^2(A+B)^2 d_{(n-2)(n-2)}(A^* + B^*)^2 \\
&\vdots \\
&\leq a^n(A+B)^n d_{00}(A^* + B^*)^n
\end{aligned}$$

where $d_{00} = d(x_1, x_0) + d(y_1, y_0) = C$. Put $P = (A+B)$ and $P^* = (A+B)^*$, so we obtain

$$\begin{aligned} d_n n &\leq a^n P^n C (P^*)^n = a^n P^n C^{\frac{1}{2}} C^{\frac{1}{2}} (P^*)^n \\ &= a^n (P^n C^{\frac{1}{2}})(C^{\frac{1}{2}}(P^*)^n) = a^n (P^n C^{\frac{1}{2}})(P^n C^{\frac{1}{2}})^*. \end{aligned}$$

Since C positive element in $\mathcal{A}$, so $C^* = C$ and $(C^{\frac{1}{2}})^* = C^{\frac{1}{2}}$ hold.

For $n+1 > m$ we get

$$\begin{aligned} & d_{nm} = d(x_{n+1}, x_m) + d(y_{n+1}, y_m) \\ & \leq a^n (P^n C^{\frac{1}{2}})(P^n C^{\frac{1}{2}})^* + a^{n-1} (P^{n-1} C^{\frac{1}{2}})(P^{n-1} C^{\frac{1}{2}})^* \\ & + ... + a^m (P^m C^{\frac{1}{2}})(P^m C^{\frac{1}{2}})^* \\ & = \sum_{k=m}^{n} a^k (P^k C^{\frac{1}{2}})(P^k C^{\frac{1}{2}})^* \\ & = \sum_{k=m}^{n} |(P^k a^{\frac{k}{2}} C^{\frac{1}{2}})|^2. \end{aligned}$$

So, we obtain

$$\begin{aligned} & \|d_{nm}\| \leq \| \sum_{k=m}^{n} |(P^k a^{\frac{k}{2}} C^{\frac{1}{2}})|^2 \| \\ & \leq \sum_{k=m}^{n} \|P^k\|^2 \|a^k C^{\frac{1}{2}}\|^2 I \\ & = \|C^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|P^k\|^2 \|a\|^{\frac{k}{2}} \\ & \leq \|C^{\frac{1}{2}}\|^2 \sum_{k=m}^{n} \|P a^{\frac{1}{4}}\|^{2k} I, (\|P^2\| \leq \|P\|^2. \end{aligned}$$

Put $Pa^{\frac{1}{4}} = Q$, then

$$\|d_{nm}\| \leq \|C^{\frac{1}{2}}\|^2 \frac{\|Q\|^{2m}}{1 - \|Q\|} I.$$

So, we have

$$\|d(x_{n+1}, x_m) + d(y_{n+1}, y_m)\| \leq \|C^{\frac{1}{2}}\|^2 \frac{\|Q\|^{2m}}{1 - \|Q\|} I$$

and

$$\lim_{m \to \infty} \|C^{\frac{1}{2}}\|^2 \frac{\|Q\|^{2m}}{1 - \|Q\|} I = 0_{\mathcal{A}}.$$

Therefore, (x_n) and (y_n) are cauchy sequences in X with respect to $\mathcal{A}$. So, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y$$

since $(\mathrm{X}, \mathcal{A}, d)$ is a completed. By using inequalities (1), we get

$$\begin{aligned}0 &\le d(T(x,y),x) \le a(d(T(x,y),x_{n+1}) + d(x_{n+1},x))\\ &= a(d(T(x,y),T(x_n,y_n)) + d(x_{n+1},x))\\ &\le a(Ad(x,x_n)A^* + Bd(y,y_n)B^* + d(x_{n+1},x)).\end{aligned}$$

We obtain

$$\lim_{n\to\infty} d(x_n,x) = 0, \lim_{n\to\infty} d(y_n,y) = 0, \lim_{n\to\infty} d(x_{n+1},x) = 0.$$

Thus,

$$\lim_{n\to\infty} T(x,y) = x, \lim_{n\to\infty} T(y,x) = y$$

in a similar way. So, (x, y) is coupled fixed point.

Now, we assume that (x', y') is another coupled fixed point of T then we obtain some of equalities as follows:

$$\begin{aligned}&d(x',x) = d(T(x',y'),T(x,y)) \le a(Ad(x',x)A^* + Bd(y',y)B^*)\\ &d(y',y) \le a(Ad(y',y)A^* + Bd(x',x)B^*).\end{aligned}$$

So,we get

$$d(x',x)+d(y',y) = a(A+B)[d(x',x)+d(y',y)](A^*+B^*) = aP[d(x',x)+d(y',y)]P^*.$$

Hence,we obtain following equality by using $\|P^*\| \le \|P\|$ since $P \in \mathcal{A}$

$$\|d(x',x) + d(y',y)\| \le a\|P\|\|d(x',x) + d(y',y)\|\|P^*\| \le \|d(x',x) + d(y',y)\|.$$

This is a contradiction, so we get

$$d(x',x) + d(y',y) = 0_{\mathcal{A}}.$$

Moreover, we obtain

$$d(x',x) = 0_{\mathcal{A}}, d(y',y) = 0_{\mathcal{A}}$$

since $d(x', x)$ and $d(y', y)$ are positive elements. So, we can easily seee that $x' = x$ and $y' = y$ satisfy. Therefore, T has a unique coupled fixed point. □

Corollary 3.2. if we choose $a = I$ in the Theorem 3.1, then T has a Unique Coupled Fixed Point in the C^*-algebra valued metric space.

Corollary 3.3. Let $(\mathrm{X}, \mathcal{A}, d)$ be a complete C^*-algebra valued b- metric space.We suppose that the mapping $T : X \times X \longrightarrow X$ satisfies

$$d(T(x,y),T(u,v)) \le A[d(x,u) + d(y,v)]A^*,$$

for $x, y, u, v \in X$ and $A \in \mathcal{A}$, where $\|A\| \le \frac{1}{2}$. Then T has a unique coupled fixed point.

References

[1] B. Blackadar, *Theory of operator C^*-algebras and Von Neumann algebras*, Springer, 2006.

[2] L.J.B. Ciric, *A generalization of Banach's contraction Principle*, Proc. Amer. Math. Soc., 45 (1974), 267-273.

[3] K.M. Das, K.V. Naik, *Common fixed theorems for commuting maps on a metric space*, Proc. Amer. Math. Soc., 77 (1979), 369-373.

[4] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, Amer. Math. Soc., Providence, 1996.

[5] L. Gajic, V. Rakocevic, *Pair of non-self mappings and common fixed points*, App. Math. Comput., 187 (2007), 999-1006.

[6] I. Gelfand, M. Naimark, *On the embedding of normed rings into the ring of operators in Hilbert space*, Math. Sb., 12 (1943), 197-213.

[7] L.G. Huang, X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl., 332 (2007), 1468-1476.

[8] G. Jungck, *Commuting maps and fixed points*, Amer Math. Monthly, 83 (1976), 261-263.

[9] Zhenhua Ma, Lining Jiang, Hongkai Sun, *C^*-algebra valued metric spaces and realated fixed point theorems*, Fixed Point Theory and thier applications, 206 (2014), 1-11.

[10] Zhenhua Ma, Lining Jiang, *C^*-algebra valued b-metric spaces and realated fixed point theorems*, Fixed Point Theory and Thier Applications, 222, 2015.

[11] G. J. Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.

[12] Ö. Özer, S. Omran, *Common fixed point in C^*-algebra b-valued metric space*, In the Application of mathematics n technical and natural scences: 8th International Conference for Promoting the Application of Mathematics in Technical and Natural Sciences (AMiTaNS'16), 1773 (1), 050005, AIP Publishing, 2016.

[13] Ö. Özer, S. Omran, *On the generalized C^*-valued metric spaces related with Banach fixed point theory*, International Journal of Advanced and Applied Sciences, 4 (2017), 35-37.

[14] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[15] V. Rakocevic, *Functional analysis*, Naucna Knjiga, Beograd, 1994.

[16] Sh. Rezapour, R. Hamlbarani, *Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl., 345 (2008), 719-724.

[17] A.E. Sayed, S. Omran, A. Asad, *Fixed point theorems in quaternion-valued metric spaces*, Abstract and Applied Analysis, 2014.

Accepted: 9.02.2018